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ON THE PERSISTENT DIFFICULTY OF DISJUNCTION

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Tell me where is fancy bred,
Or in the heart, or in the head?
Shakespeare, Merchant of Venice, III.ii.64

In intuitionistic analysis, the union of two closed subsets of Baire space $\mathcal{N}$ is not always closed. More generally, the union of a closed set and a $\mathbf{\Pi}^0_n$-set is not always $\mathbf{\Pi}^0_{n+1}$. In the proof of this fact we use the intuitionistic Borel Hierarchy Theorem.

1. THE STARTING POINT

1.1. Our subject belongs to intuitionistic analysis. We interpret logical connectives and quantifiers and set-theoretical operations constructively. If we want to prove a disjunctive statement $A \vee B$ we have to provide either a proof of A or a proof of B. If we want to prove an existential statement $\exists x \in V[A(x)]$, we have to find an element x_0 of the set V and then to perform a proof of the statement $A(x_0)$.

1.2. $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers. It is never complete and always in the state of becoming. Its elements are produced one by one, in a never ending sequence.
Baire space $\mathcal{N}$ is the set of all infinite sequences of natural numbers. Every element of $\mathcal{N}$ is a function α from $\mathbb{N}$ to $\mathbb{N}$ and in some sense an infinite and incomplete object, a kind of project to be carried out or filled in in the future; its values $\alpha(0), \alpha(1), \alpha(2), \dots$ are produced one by one, in a never ending sequence. We do not require that there exists a rule or a finitely described algorithm that defines the project, and makes that it could be left to itself, we ourselves take care that the project is continued and supply a next value every time, perhaps freely choosing one, perhaps following some more or less secret device of our own.
We use $m, n, p, \dots$ as variables over $\mathbb{N}$ and $\alpha, \beta, \gamma, \dots$ as variables over $\mathcal{N}$.

1.3. First Axiom of Countable Choice

> Let R be a subset of $\mathbb{N} \times \mathbb{N}$ and suppose that for every m there exists n such that mRn.
> Then there exists α in $\mathcal{N}$ such that, for every m, $mR\alpha(m)$.

Why? We simply choose the values of the promised sequence α one by one and do not worry about a rule that governs the process of choosing.

1.4. $\mathbb{N}^*$ is the set of all finite sequences of natural numbers. We assume that we are given a one-to-one function from $\mathbb{N}^*$ onto $\mathbb{N}$ that associates to every element $(s_0, \ldots, s_{n-1})$ of $\mathbb{N}^*$ a natural number $\langle s_0, \ldots, s_{n-1}\rangle$.
For every s, n in $\mathbb{N}$, if $s = \langle s_0, \ldots, s_{n-1}\rangle$, we say that s has *length* n.
We let $*$ be the function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ with the property that for all s, t in $\mathbb{N}$, $s * t$ is the code number of the finite sequence of natural numbers that we obtain by putting the sequence coded by t behind the sequence coded by s.
Every α in $\mathcal{N}$ splits into an infinite sequence $\alpha^0, \alpha^1, \ldots$ of elements of $\mathcal{N}$: we define, for all m, n, $\alpha^m(n) := \alpha(\langle m\rangle * n)$.
For every α in $\mathcal{N}$, m, n in $\mathbb{N}$, we define $\alpha^{m,n} := (\alpha^m)^n$.
For every α in $\mathcal{N}$, m in $\mathbb{N}$, we define $\overline{\alpha}(m) := \langle \alpha(0), \ldots, \alpha(m-1)\rangle$.
We sometimes write "$\overline{\alpha}m$" rather than "$\overline{\alpha}(m)$".
For every α in $\mathcal{N}$, s in $\mathbb{N}$, we say that α *goes through* s or that s is an *initial part* of α if and only if there exists m such that $s = \overline{\alpha}m$.
For every n in $\mathbb{N}$, we let $\underline{n}$ be the element of $\mathcal{N}$ with the constant value n.

1.5. Second Axiom of Countable Choice

> Let R be a subset of $\mathbb{N} \times \mathcal{N}$ and suppose that for every m there exists α such that $mR\alpha$.
> Then there exists α in $\mathcal{N}$ such that, for every m, $mR\alpha^m$.

Why? We again choose the values of such an α one by one. In fact we start and keep going an infinite sequence of projects for making an infinite sequence of natural numbers.

1.6. Brouwer's Continuity Principle

> Let R be a subset of $\mathbb{N} \times \mathbb{N}$ and suppose that for every α there exists n such that αRn.
> Then, given any α in $\mathcal{N}$, we may calculate m, n such that for every β, if $\overline{\beta}m = \overline{\alpha}m$, then βRn.

Why? Every α in $\mathcal{N}$ may be imagined to result from a free step-by-step-construction, and if α arises in this way, we will have completed the construction of a natural number n suitable for α after only a finite number of steps in the construction of α.

1.7. Let F be a subset of Baire space $\mathcal{N}$.
The *closure* $\overline{F}$ of F is the set of all α in $\mathcal{N}$ such that for every n there exists β in F

going through $\overline{\alpha}n$.
F will be called a *spread* if and only if the following two conditions are satisfied:

(i) $\overline{F}$ coincides with F
(ii) For every s in $\mathbb{N}$ we may decide if there exists β in F going through s or not.

1.8. Brouwer's Continuity Principle, extended version

> Let $F \subseteq \mathcal{N}$ be a spread.
> Let R be a subset of $F \times \mathcal{N}$ and suppose that for every α in F there exists n in $\mathbb{N}$ such that $\alpha R n$.
> Then, given any α in F, we may calculate m, n such that for every β in F, if $\overline{\beta}m = \overline{\alpha}m$, then $\beta R n$.

It is not difficult to derive the extended version of the Continuity Principle from the Continuity Principle itself. We have to be aware that, given any spread F, there is a continuous function r_F from $\mathcal{N}$ onto F such that for every α in F, $r_F(\alpha) = \alpha$.

1.9. We let **Fun** be the set of all γ in $\mathcal{N}$ such that for every α in $\mathcal{N}$ there exists n in $\mathbb{N}$ such that $\gamma(\overline{\alpha}n) \neq 0$.
For every γ in **Fun** and every α in $\mathcal{N}$ we calculate a natural number $\gamma(\alpha)$ as follows: let n_0 be the least natural number n such that $\gamma(\overline{\alpha}n) \neq 0$ and define $\gamma(\alpha) := \gamma(\overline{\alpha}n_0) - 1$.
In this way every γ in **Fun** acts as a code for a continuous function from $\mathcal{N}$ to $\mathbb{N}$.

1.10. Observe that for every γ in **Fun**, if $\gamma(\langle\,\rangle) = 0$, then for each m, γ^m belongs to **Fun**.
For every γ in **Fun** such that $\gamma(\langle\,\rangle) = 0$ and every α in $\mathcal{N}$ we define an element $\gamma|\alpha$ of $\mathcal{N}$ by stipulating that for each m, $(\gamma|\alpha)(m) := \gamma^m(\alpha)$. In this way every γ in **Fun** such that $\gamma(\langle\,\rangle) = 0$ acts as a code for a continuous function from $\mathcal{N}$ to $\mathcal{N}$.

2. POSITIVELY BOREL SETS

2.1. A subset C of $\mathbb{N}$ will be called a *decidable* subset of $\mathbb{N}$ if and only if for each natural number n, either $n \in C$ or $n \notin C$.
We do not require that there exists an algorithm for taking these decisions: we allow ourselves to take them one by one.

2.2. A subset X of $\mathcal{N}$ will be called *open* or $\mathbf{\Sigma}^0_1$ if and only if there exists a decidable subset C of $\mathbb{N}$ such that for every α in $\mathcal{N}$, α belongs to X if and only if some initial part $\overline{\alpha}m$ of α belongs to C. A subset X of $\mathcal{N}$ will be called *closed* or $\mathbf{\Pi}^0_1$ if and only if its complement $\mathcal{N} \backslash X$ is open.
It follows that a subset X of $\mathcal{N}$ is closed if and only if there exists a decidable subset C of $\mathcal{N}$ such that for every α in $\mathcal{N}$, α belongs to X if and only if every initial part $\overline{\alpha}m$ of α belongs to C. The complement of a closed set is not, in general, an open subset of $\mathcal{N}$.

2.3. We define a sequence $\mathbf{\Sigma}^0_2, \mathbf{\Pi}^0_2, \mathbf{\Sigma}^0_3, \mathbf{\Pi}^0_3, \ldots$ of classes of subsets of $\mathcal{N}$ by induction. Let n be a natural number. A subset X of $\mathcal{N}$ belongs to $\mathbf{\Sigma}^0_{n+1}$ if and only if there exists a sequence $Y_0, Y_1, \ldots$ of elements of $\mathbf{\Pi}^0_n$ such that $X = \bigcup\limits_{n\in\mathbb{N}} Y_n$. A subset X of $\mathcal{N}$ belongs to $\mathbf{\Pi}^0_{n+1}$ if and only if there exists a sequence $Y_0, Y_1, \ldots$ of elements of $\mathbf{\Sigma}^0_n$ such that $X = \bigcap\limits_{n\in\mathbb{N}} Y_n$.
A subset of $\mathcal{N}$ that belongs to one of the classes $\mathbf{\Sigma}^0_n$ is called a *positively Borel set of finite rank.*

2.4. Let X, Y be subsets of $\mathcal{N}$ and let γ belong to **Fun** and assume $\gamma(\langle\,\rangle) = 0$. We say that γ *reduces* X to Y if and only if, for every α, α belongs to X if and only if $\gamma|\alpha$ belongs to Y.
X *reduces to* Y, or X is *reducible to* Y, notation $X \preccurlyeq Y$, if and only if some γ in **Fun** such that $\gamma(\langle\,\rangle) = 0$ reduces X to Y.
The relation of reducibility is easily seen to be reflexive and transitive.

2.5. From now on, we use the symbol $*$ also to denote the function from $\mathbb{N} \times \mathcal{N}$ to $\mathcal{N}$ such that for all s in $\mathbb{N}$, α in $\mathcal{N}$, $s * \alpha$ is the infinite sequence that we obtain by putting the infinite sequence α behind the finite sequence coded by s.
For every s in $\mathbb{N}$ and every subset X of $\mathcal{N}$ we define $s * X := \{s * \alpha | \alpha \in X\}$. For all subsets X, Y of $\mathcal{N}$ we define $X \oplus Y := \langle 0\rangle * X \cup \langle 1\rangle * Y$.
For every sequence $X_0, X_1, X_2, \ldots$ of subsets of $\mathcal{N}$ we define
$\sum\limits_{n\in\mathbb{N}} X_n := \bigcup\limits_{n\in\mathbb{N}} \langle n\rangle * X_n$.

2.6. **Theorem**:

(i) For all subsets X, Y, Z of $\mathcal{N}$, $X \oplus Y \preccurlyeq Z$ if and only if both $X \preccurlyeq Z$ and $Y \preccurlyeq Z$.
(ii) For every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, for every subset Z of $\mathcal{N}$: $\sum\limits_{n\in\mathbb{N}} X_n \preccurlyeq Z$ if and only if for each n, $X_n \preccurlyeq Z$.

Proof: The proof is straightforward. We have to use the Second Axiom of Countable Choice in the proof of (ii). ⊠

2.7. For all subsets X, Y of $\mathcal{N}$ we let $D(X, Y)$, the *disjunction* of X and Y, be the set of all α in $\mathcal{N}$ such that either α^0 belongs to X or α^1 belongs to Y. For every subset X of $\mathcal{N}$, for every natural number n, we let $D^n(X)$, the *n-fold disjunction* of X, be the set of all α in X such that for some $j < n$, α^j belongs to X. Observe that $D^0(X)$ is the empty set.
For every subset X of $\mathcal{N}$, we let $D^\omega(X)$, the *ω-fold disjunction* of X, be the set of all α in $\mathcal{N}$ such that for some j, α^j belongs to X. For every subset X of $\mathcal{N}$, we let $C^\omega(X)$, the *ω-fold conjunction* of X, be the set of all α in $\mathcal{N}$ such that for every j, α^j belongs to X.

2.8. **Theorem**:

(i) For all subsets Z, X, Y of $\mathcal{N}$, Z reduces to $D(X, Y)$ if and only if there exist subsets Z_0, Z_1 of Z such that $Z = Z_0 \cup Z_1$ and both Z_0 reduces to X and Z_1 reduces to Y.

(ii) For all subsets Z, X of $\mathcal{N}$, Z reduces to $D^\omega(X)$ if and only if there exists a sequence $Z_0, Z_1, \ldots$ of subsets of Z such that $Z = \bigcup\limits_{n \in \mathbb{N}} Z_n$ and each Z_n reduces to X.

(iii) For all subsets Z, X of $\mathcal{N}$, Z reduces to $C^\omega(X)$ if and only if there exists a sequence $Z_0, Z_1, \ldots$ of subsets of $\mathcal{N}$ such that $Z = \bigcap\limits_{n \in \mathbb{N}} Z_n$ and each Z_n reduces to X.

Proof: The proof is straightforward. We have to use the Second Axiom of Countable Choice in the proof of (ii) and (iii). ⊠

2.9. We define a sequence (A_1, E_1), $(A_2, E_2), \ldots$ of pairs of subsets of $\mathcal{N}$.
$A_1 := \{\alpha | \alpha \in \mathcal{N} | \forall n[\alpha(n) = 0]\}$ and $E_1 := \{\alpha | \alpha \in \mathcal{N} | \exists n[\alpha(n) \neq 0]\}$, and, for each n, $A_{n+1} := C^\omega(E_n)$ and $E_{n+1} := D^\omega(A_n)$.

2.10. **Theorem**:

For each n, E_n belongs to $\mathbf{\Sigma}^0_n$ and every $\mathbf{\Sigma}^0_n$-set reduces to E_n and A_n belongs to $\mathbf{\Pi}^0_n$ and every $\mathbf{\Pi}^0_n$-set reduces to A_n.

Proof: The proof is straightforward. ⊠

Because of this Theorem, we want to call the pairs $(A_1, E_1), (A_2, E_2), \ldots$ the *leading pairs of the finite Borel Hierarchy.*

3. THINKING STRATEGICALLY

3.1. Consider the pair (A_4, E_4). For every α in $\mathcal{N}$ we devise a four-move-game $G_4(\alpha)$ for players I,II. It is a game of perfect information, so every player knows at every moment the moves that have been played so far.

I chooses s_0 $\quad$ I chooses s_2

II chooses s_1 $\quad$ II chooses s_3

I wins the play $(s_0, \ldots, s_3)$ if $\alpha(\langle s_0, \ldots, s_3 \rangle) = 0$
II wins the play $(s_0, \ldots, s_3)$ if $\alpha(\langle s_0, \ldots, s_3 \rangle) \neq 0$.

Using the First Axiom of Countable Choice, we observe that for every α, $E_4(\alpha)$ if and only if $\exists s_0 \forall s_1 \exists s_2 \forall s_3[\alpha(\langle s_0, \dots, s_3\rangle) = 0]$ if and only if $\exists\gamma\forall s_1\forall s_3[\alpha(\langle\gamma(\langle\rangle), s_1, \gamma(\langle s_1\rangle), s_3\rangle) = 0]$, that is, player I has a winning strategy in the game $G_4(\alpha)$.
Similarly, for every α, $A_4(\alpha)$ if and only if $\forall s_0\exists s_1\forall s_2\exists s_3[\alpha(\langle s_0, \dots, s_3\rangle) \neq 0]$ if and only if $\exists\gamma\forall s_0\forall s_2[\alpha(\langle s_0, \gamma(\langle s_0\rangle), s_2, \gamma(\langle s_0, s_2\rangle) \neq 0]$, that is, player II has a winning strategy in the game $G_4(\alpha)$.
One may guess how to define, for every positive number n, the game $G_n(\alpha)$.
We now define, for every natural number $s = \langle s_0, \dots, s_{n-1}\rangle$ and every γ in $\mathcal{N}$: γ *I-governs* s,or s *I-obeys* γ if and only if for each $i < n$, if i is even, then $s_i = \gamma(\langle s_1, s_3, \dots, s_{2k+1}\rangle)$ where $2k+1$ equals $i-1$, and γ *II-governs* s, or s *II-obeys* γ if and only if for each $i < n$, if i is odd, then $s_i = \gamma(\langle s_0, s_2, \dots, s_{2k}\rangle)$, where $2k$ equals $i-1$.

3.2. **Theorem**:

(i) For every positive even natural number n, for every α, $E_n(\alpha)$ if and only if there exists γ such that for every s I-obeying γ of length n, $\alpha(s) = 0$, and $A_n(\alpha)$ if and only if there exists γ such that for every s II-obeying γ of length n, $\alpha(s) \neq 0$.

(ii) For every odd number n, for every α, $E_n(\alpha)$ if and only if there exists γ such that for every s I-obeying γ of length n, $\alpha(s) \neq 0$, and $A_n(\alpha)$ if and only if there exists γ such that for every s II-obeying γ of length n, $\alpha(s) = 0$.

Proof: The proof is straightforward. ⊠

3.3 For every α, γ in $\mathcal{N}$ and n in $\mathbb{N}$ we define two elements of $\mathcal{N}$, $\mathrm{Corr}_I^n(\gamma, \alpha)$ and $\mathrm{Corr}_{II}^n(\gamma, \alpha)$, as follows. For every s in $\mathbb{N}$, if s does not have length n or does not I-obey γ, then $\big(\mathrm{Corr}_I^n(\gamma, \alpha)\big)(s) := \alpha(s)$, but if s has length n and I-obeys γ, then $\big(\mathrm{Corr}_I^n(\gamma, \alpha)\big)(s)$ equals 0 if n is even, and 1 if both n is odd and $\alpha(s) = 0$, and $\alpha(s)$ otherwise. Similarly, for every s in $\mathbb{N}$, if s does not have length n or does not II-obey γ, then $\big(\mathrm{Corr}_I^n(\gamma, \alpha)\big)(s) := \alpha(s)$, but if s has length n and II-obeys γ, then $\big(\mathrm{Corr}_{II}^n(\gamma, \alpha)\big)(s)$ equals 0 if n is odd, and 1 if both n is even and $\alpha(s) = 0$, and $\alpha(s)$ otherwise. We might pronounce "$\mathrm{Corr}_I^n(\gamma, \alpha)$" as "$\alpha$-as-corrected-according-to-γ-as-a-strategy-for-player-I-in-the-game G_n", quite a mouthful.

3.4. **Theorem**:

For every n, for every α, $E_n(\alpha)$ if and only if there exists γ in $\mathcal{N}$ such that $\alpha = \mathrm{Corr}_I^n(\gamma, \alpha)$, and $A_n(\alpha)$ if and only if there exists γ in $\mathcal{N}$ such that $\alpha = \mathrm{Corr}_{II}^n(\gamma, \alpha)$.

Proof: The proof is straightforward. ⊠

4. A REMARK ON CONVERGENT SEQUENCES

4.1. Let A be a subset of the set $\mathbb{N}$ of natural numbers. We say that *almost all* natural numbers belong to A if and only if there exists a natural number n such that every natural number m greater than n belongs to A.

4.2. Let α, β be elements of $\mathcal{N}$. We say that α *lies apart from* β, notation $\alpha \# \beta$, if and only if there exists n such that $\alpha(n) \neq \beta(n)$.
Let $\alpha_0, \alpha_1, \ldots$ be a sequence of elements of $\mathcal{N}$, and let β be an element of $\mathcal{N}$.
We say that the sequence $\alpha_0, \alpha_1, \ldots$ *converges* to β if and only if for every n, for almost every m, $\overline{\beta}n = \overline{\alpha_m}n$. We say that the sequence $\alpha_0, \alpha_1, \ldots$ *discernibly converges* to β if and only if, in addition, for all m, n, if $m \neq n$, then α_m is lying apart from α_n and from β.
We say that the sequence $\alpha_0, \alpha_1, \ldots$ is (*discernibly*) *convergent* if and only if there exists β in $\mathcal{N}$ such that the sequence $\alpha_0, \alpha_1, \ldots$ (discernibly) converges to β.

4.3. Let $\alpha_0, \alpha_1, \ldots$ be a convergent sequence of elements of $\mathcal{N}$.
Consider $X := \{\alpha_0, \alpha_1, \ldots\}$ and observe that the closure $\overline{X}$ of X is a spread.
If $\alpha_0, \alpha_1, \ldots$ discernibly converges to β, then one may find, for every δ in $\overline{X}$ that lies apart from β, some n such that $\alpha_n = \delta$.

4.4. **Lemma**:

> Let $\alpha_0, \alpha_1, \ldots$ be a convergent sequence of elements of X and consider $X := \{\alpha_0, \alpha_1, \ldots\}$.
> Suppose that P, Q are subsets of the closure $\overline{X}$ of X such that every element of $\overline{X}$ belongs either to P or to Q.
> Then for almost every n, α_n belongs to P, or for almost every n, α_n belongs to Q.

Proof: Calculate β in $\mathcal{N}$ such that for every n, for almost every m, $\overline{\beta}n = \overline{\alpha_m}n$.
Observe that β belongs to the spread $\overline{X}$.
Using the Continuity Principle, we find n such that either every α in $\overline{X}$ passing through $\overline{\beta}n$ belongs to P or every α in $\overline{X}$ passing through $\overline{\beta}n$ belongs to Q.
Observe that for almost every m, α_m passes through $\overline{\beta}n$. ⊠

4.5. Lemma 4.4 will be used in the sequel.
The following observation is not immediately needed. Let $\alpha_0, \alpha_1, \ldots$ be a convergent sequence of elements of $\mathcal{N}$ and consider $X := \{\alpha_0, \alpha_1, \ldots\}$. If X coincides with $\overline{X}$, there will exist n such that for almost every m, $\alpha_m = \alpha_n$.

5. THE FIRST STEP

5.1. We want to show that the set $D(A_1, A_1)$ does not reduce to the set A_1. To this end, we define a sequence $\alpha_0, \alpha_1, \ldots$ of elements of $\mathcal{N}$, as follows. For each n, $(\alpha_{2n})^0 := (\alpha_{2n+1})^1 := \underline{0}$ and $(\alpha_{2n})^1 := (\alpha_{2n+1})^0 := \overline{\underline{0}}n * \underline{1}$ and for each k, if there is no p such that $k = \langle 0 \rangle * p$ or $k = \langle 1 \rangle * p$, then $\alpha_{2n}(k) := \alpha_{2n+1}(k) := 0$.
Observe that the sequence $\alpha_0, \alpha_1, \ldots$ discernibly converges to $\underline{0}$ and that for each n, $(\alpha_{2n})^0$ and $(\alpha_{2n+1})^1$ belong to A_1 but $(\alpha_{2n})^1$ and $(\alpha_{2n+1})^0$ do not.

5.2. **Theorem**: The set $D(A_1, A_1)$ does not reduce to the set A_1.

Proof: Assume γ belongs to **Fun**, $\gamma(\langle\,\rangle) = 0$ and γ reduces $D(A_1, A_1)$ to A_1, and therefore, for every α, $\alpha^0 = \underline{0}$ or $\alpha^1 = \underline{0}$ if and only if $\gamma|\alpha = \underline{0}$. Consider the sequence $\alpha_0, \alpha_1, \ldots$ as introduced in Section 5.1, and define $X := \{\alpha_0, \alpha_1, \ldots\}$. We claim that γ maps the closure $\overline{X}$ of X into A_1. For, let β belong to $\overline{X}$ and n to $\mathbb{N}$. Calculate m such that for every α, if $\overline{\alpha}m = \overline{\underline{0}}m$, then $(\gamma|\alpha)(n) = (\gamma|\underline{0})(n) = 0$. Now distinguish two cases. *Either* $\overline{\beta}m = \overline{\underline{0}}m$ and $(\gamma|\beta)(n) = 0$, *or* $\overline{\beta}m \neq \overline{\underline{0}}m$, therefore there exists i such that $\beta = \alpha_i$ and β belongs to $D(A_1, A_1)$, therefore $\gamma|\beta = \underline{0}$, in particular $(\gamma|\beta)(n) = 0$.
We may conclude that $\overline{X}$ forms part of $D(A_1, A_1)$, therefore by Lemma 4.4, either for almost all n, $(\alpha_n)^0 = \underline{0}$ or for almost all n, $(\alpha_n)^1 = \underline{0}$.
Contradiction. ⊠

We may prove more.

5.3. **Theorem**: The set $D(A_1, A_1)$ does not reduce to the set A_2.

Proof: Assume γ belongs to **Fun**, $\gamma(\langle\,\rangle) = 0$ and reduces $D(A_1, A_1)$ to A_2, that is, for every α, $\alpha^0 = \underline{0}$ or $\alpha^1 = \underline{0}$ if and only if $\gamma|\alpha$ belongs to A_2. Consider the sequence $\alpha_0, \alpha_1, \ldots$ as introduced in Section 5.1 and define $X := \{\alpha_0, \alpha_1, \ldots\}$. We claim that γ maps the closure $\overline{X}$ of X into A_2. For, let β belong to $\overline{X}$ and n to $\mathbb{N}$. Calculate p such that $(\gamma|\underline{0})^n(p) \neq 0$. Calculate m such that for every α, if $\overline{\alpha}m = \overline{\underline{0}}m$, then $(\gamma|\alpha)^n(p) = (\gamma|\underline{0})^n(p) \neq 0$. Now distinguish two cases. *Either* $\overline{\beta}m = \overline{\underline{0}}m$ and $(\gamma|\alpha)^n(p) \neq 0$, *or* $\overline{\beta}m \neq \overline{\underline{0}}m$. In the latter case find i such that $\beta = \alpha^i$ and observe: β belongs to $D(A_1, A_1)$, therefore $\gamma|\beta$ belongs to A_2, in particular there exists q such that $(\gamma|\beta)^n(q) \neq 0$.
We may conclude that $\overline{X}$ forms part of $D(A_1, A_1)$ and obtain a contradiction, as in the proof of Theorem 5.2. ⊠

6. THE SECOND STEP

6.1. **Lemma**:

> Suppose that γ belongs to **Fun**, $\gamma(\langle\,\rangle) = 0$, and γ maps A_2 into E_2. Then for every α in A_2 there exists m, n such that for every β, if $\overline{\beta}m = \overline{\alpha}m$, then $(\gamma|\beta)^n = \underline{0}$.

Proof: Suppose that α belongs to A_2 and determine δ such that $\alpha = \mathrm{Corr}^2_{II}(\delta, \alpha)$. Apply the Continuity Principle and calculate p, n such that for all ζ, β, if $\overline{\zeta}p = \overline{\delta}p$ and $\overline{\beta}p = \overline{\alpha}p$, then $\left(\mathrm{Corr}^2_{II}(\zeta, \beta)\right)^n = \underline{0}$.
Calculate $m > p$ such that for every β, if β belongs to A_2 and $\overline{\beta}m = \overline{\alpha}m$, then there exists ζ such that $\overline{\zeta}p = \overline{\delta}p$ and $\beta = \mathrm{Corr}^2_{II}(\zeta, \beta)$.
Now assume β belongs to $\mathcal{N}$, not necessarily to A_2, and $\overline{\beta}m = \overline{\alpha}m$. Observe that for every q there exists a member of A_2 passing through $\overline{\beta}q$, therefore $(\gamma|\beta)^n$ must be $\underline{0}$. ⊠

We formulate an extension of this Lemma.

6.2. **Lemma**:

> Suppose that γ belongs to **Fun**, and $\gamma(\langle\,\rangle) = 0$.
> Suppose also that p is a natural number and that for every α, if α^p belongs to A_2, then $\gamma|\alpha$ belongs to E_2.
> Then for every α such that α^p belongs to A_2 there exist m, n such that for every β, if $\overline{\beta}m = \overline{\alpha}m$, then $(\gamma|\beta)^n = \underline{0}$.

Proof: The proof is almost the same as the proof of Lemma 6.1. ⊠

6.3. We define a sequence $\alpha_0, \alpha_1, \ldots$ of elements of $\mathcal{N}$, as follows.
For each n, $(\alpha_{2n})^0 := \underline{0}$ and $(\alpha_{2n})^1 := \overline{\underline{1}}n * \underline{0}$, and $(\alpha_{2n+1})^0 := \overline{\underline{0}}n * \underline{1}$ and $(\alpha_{2n+1})^1 := \underline{1}$, and, for each k, if there is no p such that $k = \langle 0\rangle * p$ or $k = \langle 1\rangle * p$, then $\alpha_{2n}(k) = \alpha_{2n+1}(k) = 0$. Let δ be the element of $\mathcal{N}$ such that $\delta^0 = \underline{0}$ and $\delta^1 = \underline{1}$ and, for each k, if there is no p such that $k = \langle 0\rangle * p$ or $k = \langle 1\rangle * p$, then $\delta(k) = 0$.
Observe that the sequence $\alpha_0, \alpha_1, \ldots$ discernibly converges to δ.
Furthermore, δ^0 belongs to A_1 and δ^1 belongs to A_2 and for each i, $(\alpha_i)^0$ belongs to A_1 if and only if i is even and $(\alpha_i)^1$ belongs to A_2 if and only if i is odd.

6.4. **Theorem**: The set $D(A_1, A_2)$ does not reduce to the set A_3.

Proof: Assume γ belongs to **Fun**, $\gamma(\langle\,\rangle) = 0$ and γ reduces $D(A_1, A_2)$ to A_3, that is, for every α, $\alpha^0 = \underline{0}$ or α^1 belongs to A_2 if and only if $\gamma|\alpha$ belongs to A_3.
Consider the sequence $\alpha_0, \alpha_1, \ldots$ as introduced in Section 6.3 and define $X := \{\alpha_0, \alpha_1, \ldots\}$. We claim that γ maps the closure $\overline{X}$ of the set X into the set A_3.
For, let β belong to $\overline{X}$ and n to $\mathbb{N}$.
Let δ be the element of $\mathcal{N}$ such that $\alpha_0, \alpha_1, \ldots$ converges to δ. Using Lemma 6.2, we calculate m, p such that for every α, if $\overline{\alpha}m = \overline{\delta}m$, then $(\gamma|\alpha)^{n,p} = \underline{0}$. Now distinguish two cases.
Either $\overline{\beta}m = \overline{\delta}m$ and $(\gamma|\alpha)^{n,p} = \underline{0}$ *or* $\overline{\beta}m \neq \overline{\delta}m$. In the latter case, calculate i such that $\beta = \alpha_i$ and observe: β belongs to $D(A_1, A_2)$, therefore $\gamma|\beta$ belongs to A_3, in particular $(\gamma|\beta)^n$ belongs to E_2 and there exists q such that $(\gamma|\beta)^{n,q} = \underline{0}$.
We may conclude that $\overline{X}$ forms part of $D(A_1, A_2)$, therefore, for almost all i, $(\alpha_i)^0$ belongs to A_1 or, for almost all i, $(\alpha_i)^1$ belongs to A_2. Contradiction. ⊠

7. THE THIRD STEP

The third step is more difficult than the two previous ones.

7.1. **Lemma**:

> Suppose that γ belongs to **Fun**, $\gamma(\langle\,\rangle) = 0$ and γ maps A_3 into E_3. Then for every α in A_3 there exist m, n such that for every β in A_3, if $\beta(\langle\,\rangle) = \alpha(\langle\,\rangle)$ and for all $i < m$, $\beta^i = \alpha^i$, then $(\gamma|\beta)^n$ belongs to A_2.

Proof: Suppose that α belongs to A_3 and determine δ such that $\alpha = \mathrm{Corr}^3_{II}(\delta, \alpha)$. Apply the Continuity Principle and calculate p, n such that for all ζ, β, if $\overline{\zeta}p = \overline{\delta}p$ and $\overline{\beta}p = \overline{\alpha}p$, then $\left(\gamma|\mathrm{Corr}^3_{II}(\zeta, \beta)\right)^n$ belongs to A_2. Determine m in such a way that for every β in A_3, if $\beta(\langle\,\rangle) = \alpha(\langle\,\rangle)$ and for all $i < m$, $\beta^i = \alpha^i$, then $\overline{\beta}p = \overline{\alpha}p$ and there exists ζ such that $\overline{\zeta}p = \overline{\delta}p$ and $\beta = \mathrm{Corr}^3_{II}(\zeta, \beta)$. It will be clear that m, n satisfy our purposes. ⊠

7.2. **Lemma**:

> Suppose that γ belongs to **Fun**, and $\gamma(\langle\,\rangle) = 0$.
> Suppose also that p is a natural number and that for every α, if α^p belongs to A_3, then $\gamma|\alpha$ belongs to E_2.
> Then for every α such that α^p belongs to A_3, there exist m, n such that for every β, if β^p belongs to A_3, and $\overline{\beta}m = \overline{\alpha}m$ and for all $i < m$, $\beta^{p,i} = \alpha^{p,i}$, then $(\gamma|\beta)^n$ belongs to E_2.

Proof: The proof follows the pattern of the proof of Lemma 7.1 and is left to the reader. ⊠

7.3. **Theorem**:

> Suppose that γ belongs to **Fun**, and $\gamma(\langle\,\rangle) = 0$.
>
> (i) If γ maps E_2 into A_2, there exists α such that both α itself and $\gamma|\alpha$ belong to A_2.
> (ii) If γ maps A_3 into A_2, there exists α such that α itself belongs to E_3 and $\gamma|\alpha$ belongs to A_2.

Proof:

(i) Suppose that γ belongs to **Fun**, $\gamma(\langle\,\rangle) = 0$ and γ maps E_2 into A_2. Observe that for every n, for every α, if $\alpha^n = \underline{0}$, then α belongs to E_2, therefore $\gamma|\alpha$ belongs to A_2 and $(\gamma|\alpha)^n$ belongs to E_1, so there exists p such that $(\gamma|\alpha)^n(p) \neq 0$, and there exists q such that for every β, if $\overline{\beta}q = \overline{\alpha}q$, then $(\gamma|\beta)^n(p) = (\gamma|\alpha)^n(p) \neq 0$, so $(\gamma|\beta)^n$ belongs to E_1.
We now define a sequence $\alpha_0, \alpha_1, \ldots$ of elements of $\mathcal{N}$ and a sequence $q_0, q_1, \ldots$ of natural numbers with the following properties:

for each n, $q_n < q_{n+1}$, and $\overline{\alpha_n} q_n = \overline{\alpha_{n+1}} q_n$ and $(\alpha_n)^n = \underline{0}$ and for every $j < n$ there exists k such that $\langle j, k \rangle < q_n$ and $\alpha_n(\langle j, k \rangle) \neq 0$, and for every β, if $\overline{\beta} q_n = \overline{\alpha}_n q_n$, then $(\gamma | \alpha)^n$ belongs to E_1.

Let α be the element of $\mathcal{N}$ such that for every n, $\overline{\alpha} q_n = \overline{\alpha}_n q_n$.
It will be clear that both α and $\gamma | \alpha$ belong to A_2.

(ii) Let δ be an element of **Fun** such that $\delta(\langle\,\rangle) = 0$ and for every n, for every α, $(\delta | \alpha)^n = \alpha$. Observe that δ reduces E_2 to A_3. Now assume that γ belongs to **Fun**, $\gamma(\langle\,\rangle) = 0$ and γ maps A_3 into A_2. Determine ζ in **Fun** such that $\zeta(\langle\,\rangle) = 0$ and for every α, $\zeta | \alpha = \gamma | (\delta | \alpha)$. ζ maps E_2 into A_2, so we may determine α such that both α and $\zeta | \alpha$ belong to A_2. Observe that $\delta | \alpha$ belongs to E_3 and $\gamma | (\delta | \alpha)$ belongs to A_2. ⊠

7.4. **Theorem**: The set $D(A_1, A_3)$ does not reduce to the set A_4.

Proof: Suppose that γ belongs to **Fun**, $\gamma(\langle\,\rangle) = 0$ and γ reduces the set $D(A_1, A_3)$ to the set E_4, that is, for every α, $\alpha^0 = \underline{0}$ or α^1 belongs to A_3 if and only if $\gamma | \alpha$ belongs to A_4.
Observe that $\underline{0}$ belongs both to A_1 and to A_3 and also to $D(A_1, A_3)$.
Using Lemma 7.2 we calculate an infinite sequence $m_0, n_0, m_1, n_1, \ldots$ of natural numbers such that for each k, for every β, if $\overline{\beta} m_k = \overline{\underline{0}} m_k$ and for every $i < m_k$, $\beta^{1,i} = \underline{0}$ and β^1 belongs to A_3, then $(\gamma | \beta)^{k, n_k}$ belongs to A_2. We take care that $m_0 < m_1 < \cdots$.
We now define a sequence $\alpha_0, \alpha_1, \ldots$ of elements of $\mathcal{N}$.
We first define $\alpha_0, \alpha_2, \ldots$. For each k, $(\alpha_{2k})^0 := \overline{\underline{0}} m_k * \underline{1}$ and for every ℓ, if there is no p such that $\ell = \langle 0 \rangle * p$, then $\alpha_{2k}(\ell) := 0$.
Observe that $(\alpha_{2k})^0$ belongs to E_1 and therefore not to A_1, whereas $(\alpha_{2k})^1$ belongs to A_3.
We now define $\alpha_1, \alpha_3, \ldots$. Let k be a natural number. We construct δ in **Fun** such that $\delta(\langle\,\rangle) = 0$ and for every β, $(\delta | \beta)^2 = \overline{\underline{0}} m_k * \underline{1}$, and for every $i < m_k$, $(\delta | \beta)^{1,i} = \underline{0}$ and for every i, $(\delta | \beta)^{1, m_k + i} := \beta^i$, and for every ℓ, if there is no p such that $\ell = \langle 0 \rangle * p$ or $\ell = \langle 2 \rangle * p$, then $(\delta | \beta)(\ell) = 0$.
Observe that for every β, if β belongs to A_3, then $(\delta | \beta)^1$ belongs to A_3 and $\left(\gamma | (\delta | \beta)\right)^{k, n_k}$ belongs to A_2.
Using Lemma 7.3 we determine β_k in such a way that β_k belongs to E_3 and $\left(\gamma | (\delta | \beta)\right)^{k, n_k}$ belongs to A_2.
We now define: $\alpha_{2k+1} := \delta | \beta_k$ and remark that $(\alpha_{2k+1})^0$ belongs to A_1 whereas $(\alpha_{2k+1})^1$ belongs to E_3 and therefore nor to A_3.
Observe that the sequence $\alpha_0, \alpha_1, \ldots$ discernibly converges to $\underline{0}$. Remark that for every i, k, if $i > 2k$, then $(\gamma | \alpha_i)^{k, n_k}$ belongs to A_2. Now consider $X := \{\alpha_0, \alpha_1, \ldots\}$. We claim that γ maps the closure $\overline{X}$ of X into A_4. We prove this claim as follows. Let k belong to $\mathbb{N}$ and β to $\overline{X}$. We show that $(\gamma | \beta)^k$ belongs to E_3. We distinguish two cases. *Either* there exists $i < 2k$ such that $\beta = \alpha_i$, *or* for every $i \leq 2k$ β lies apart from α_i.
In the first case β belongs to $D(A_1, A_3)$ and $\gamma | \beta$ belongs to A_4, in particular $(\gamma | \beta)^k$ belongs to E_3. In the second case $(\gamma | \beta)^{k, n_k}$ must belong to A_2. For, let i belong to $\mathbb{N}$ and determine p such that $(\gamma | \underline{0})(\langle k, n_k, i, p \rangle) = 0$.

We again distinguish two cases. *Either* $(\gamma|\beta)(\langle k, n_k, i, p\rangle) = (\gamma|\underline{0})(\langle k, n_k, i, p\rangle)$,*or* β lies apart from $\underline{0}$ and there exists $i > 2k$ such that $\beta = \alpha_i$, therefore $(\gamma|\beta)^{k,n_k}$ belongs to A_2, in particular there exists q such that $(\gamma|\beta)(\langle k, n_k, i, q\rangle) = 0$.
We may conclude that $\overline{X}$ forms part of $D(A_1, A_3)$ and this, as in earlier such cases, leads to a contradiction, by Lemma 4.4. ⊠

8. WALKING ON

We now sketch how to extend the argument from Section 7 to the higher levels of the finite Borel Hierarchy. The following Theorem, an extension of Theorem 7.2, is our main tool.

8.1. **Theorem**: (*Finite Borel Hierarchy Theorem*)

> For each positive n, for each γ in **Fun** such that $\gamma(\langle\,\rangle) = 0$, if γ maps E_n into A_n, then γ maps also some member of A_n into A_n.

A proof of this Theorem may be found in Veldman 1981 or Veldman 200?
It follows from the Theorem that for each γ in **Fun**, if $\gamma(\langle\,\rangle) = 0$ and γ maps A_{n+1} into A_n, then there exists α such that α itself belongs to E_{n+1} and $\gamma|\alpha$ belongs to A_n.
Now assume that we find a positive n and some γ in **Fun** such that $\gamma(\langle\,\rangle) = 0$ and γ maps $D(A_1, A_n)$ into A_{n+1}.
We consider the case that n is odd, but the case n is even is not much different. Observe that $\underline{0}$ belongs both to A_1 and to A_n and also to $D(A_1, A_n)$. We build an infinite sequence $\alpha_0, \alpha_1, \ldots$ of elements of $\mathcal{N}$, discernibly converging to $\underline{0}$ and having the following properties:

(i) For each i, $(\alpha_i)^0$ belongs to A_1 if and only if i is odd, and $(\alpha_i)^1$ belongs to A_n if and only if i is even.
(ii) There exists δ such that $\gamma|\underline{0} = \mathrm{Corr}_{II}^{n+1}(\delta, \gamma|\underline{0})$ and for each i,there exists ζ passing through $\overline{\delta}i$ such that $\gamma|\alpha_i = \mathrm{Corr}_{II}^{n+1}(\zeta, \gamma|\alpha_i)$.

Theorem 8.1 plays an important role in the definition of $\alpha_1, \alpha_3, \ldots$.
Consider $X := \{\alpha_0, \alpha_1, \ldots\}$. Using (ii) one may prove that γ maps the closure $\overline{X}$ of X into A_{n+1}; one then concludes that $\overline{X}$ forms part of $D(A_1, A_{n+1})$ and, using Lemma 4.4, obtains a contradiction.
The reader will hopefully trust now the following result.

8.2. **Theorem**: For each n, the set $D(A_1, A_n)$ does not reduce to the set A_{n+1}.

9. CONCLUDING REMARKS

9.1. Let X, Y be subsets of $\mathcal{N}$. We say that X *strictly reduces* to Y, notation $X \prec Y$, if and only if X reduces to Y but Y does not reduce to X.

9.2. Starting from the fact that the set $D(A_1, A_1)$ does not reduce to the set A_1 one may show that for each n, the set $D^{n+1}(A_1)$ does not reduce to the set $D^n(A_1)$. We thus obtain a strictly increasing sequence

$$A_1 \prec D^2(A_1) \prec D^3(A_1) \prec \cdots .$$

Now define $B := \sum_{n\in\mathbb{N}} D^n(A_1)$. As we saw in Theorem 2.6, B is the $\preccurlyeq$-least-upper-bound of the sequence $A_1, D^2(A_1), D^3(A_1), \ldots$. One may prove that both $D(A_1, B)$ and $D(B, B)$ reduce to B. For this reason, one might call the set B *disjunctively closed.*

9.3. Other upper bounds for the sequence of the sequence $A_1, D^2(A_1), D^3(A_1)\ldots$ are not disjunctively closed. Consider for instance $C := \bigcup_{n\in\mathbb{N}} \overline{0}n * \langle 1\rangle * D^n(A_1)$. One may show that the set $D(A_1, C)$ does not reduce to the set C, and, more generally, that for each n, the set $D(D^{n+1}(A_1), C)$ does not reduce to the set $D(D^n(A_1), C)$. A fortiori, the set $D(C, C)$ does not reduce to the set C.

9.4. The sets described in Sections 9.2 and 9.3 belong to the class $\mathbf{\Sigma}^0_2$. Starting from Theorem 8.2 one may establish similar results for sets from higher classes. For instance, for each n, the set $D(D^{n+1}(A_1), A_2)$ does not reduce to the set $D(D^n(A_1), A_2)$, and the set $D^{n+1}(A_2)$ does not reduce to the set $D^n(A_2)$, and the set $D(A_2, A_3)$ does not reduce to the set $D(A_1, A_3)$. One may go on and on.

9.5. Theorems 8.1 and 8.2 generalize to the transfinite levels of the Borel Hierarchy, see Veldman 200?

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